Probabilistic assessment of the organization of tournaments and examinations using paired comparisons

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Abstract

In this paper a criteria of comparison different tournament organization systems in sporting contests is offered, the criteria uses a probability of winning the fairly strongest player. Two probabilistic models have been analyzed. Calculating formulas for estimating of that probability and probability density of score points by one or another player were obtained. Gotten results also provide an order of objects presenting to experts in organization of examination by paired comparison. An analytical estimation of probability of tournament results (or pared comparison) was obtained, it allows in many cases to avoid of time-consuming procedure of sorting out of possible variants.

Keywords: ***tournament organization, probability, paired comparison***

1 Introduction

The emergence and development of probability theory is largely related to the need for analysis of games (see [1]). A wide literature is devoted to probabilistic methods for estimating the solution of combinatorial problems (see [2]). Below we consider the formulation and analysis of the solution of such a problem as extremal task, the criterion of which is defined on the set of possible systems for processing the results of pairwise comparisons.

For many types of games, the goal is to identify the relative strength of players, and the organization of tournaments can be different for different competitions, different sports, etc. In some cases, the tournament is held in a circular system, in others - in the cup or "Swiss". In some cases, players are pre-divided into groups with subsequent drawing of the championship between the winners of groups, etc. At the same time, the conclusion about the winner in any tournament organization is based on the results of paired comparisons [3]. In connection with these circumstances, questions arise: how to evaluate the organization of the tournament; what should be the minimum number of games in order to identify the strongest with a given probability for a given number of players; what is the probability that, as a result of the tournament, the order of the places occupied by players coincides with their actual "skill"?

The number of players and the total number of games will be considered as given. Since the purpose of the tournament is to identify the strongest player, it is intuitively clear that the tournament should be organized so that the number of meetings between players close in skill is greater, and the number of meetings in which the result can be predicted (with probability close to unity) is less. To analyze the scheme of the tournament, we use the probabilistic model of the outcome of the meeting of two players, and the probability of a particular outcome of the meeting should depend on their "skill". This indicator can be introduced in different ways: as the probability of being of a player in one or another of its possible states, as the average value of some random indicator, etc.

An attempt is made to analyze analytically and numerically the various systems of the tournament. At the same time, we initially use the simplest probabilistic model of the player and model of the rule for calculating points. Then we generalize the consideration to a model closer to reality. The structure of the tournament is better, the better for the given total number of games the probability that the player with the greatest "skill" will be the winner (he will score the number of points not less than any other participant by the results of the tournament).

2 Discrete distribution of player's state and density of probability distribution of the number of points

Let the state ξ_j of the j-th player be a random discrete value, taking the value 1 with probability s_j and zero with probability $1-s_j$. The value s_j will be called the skill of the j-th player. The states of the players are independent of each other. The total number of players will be denoted by *M*.

The tournament is a sequence of paired comparisons (games) in which players' states are compared. If the j-th player's state is more than the k-th player, he gets two points, and his opponent is zero, if the states are the same, then everyone gets a point. Players are ordered in such a way that $s_1 > s_2 > ...s_M$. The result of the game, and therefore the number of points r collected in it by each participant is a random variable characterized by the probability distribution density $P(r)$ determined on the set of points $r = 0,1,2$. The probabilistic nature of the results of each game leads to random errors. Tournament can be considered as a filter separating the useful signal (a priori distribution of skills) from interference. This filter is all the better, the closer the final placement of players to the a priori alignment of their skills.

Let's order players in value s_j in such a way that $s_1 > s_2 > s_3 > \ldots$. The result of the lottery will be called ideal if the places occupied by the players $r_1 \geq r_2 \geq r_3 \ldots$, i.e. if the ranking, determined by the number of points scored and the system of lottery, corresponds to the distribution of skills. The result of the lottery will be called correct if the player with the skill s_1 was in the first place or shared it with other players, i.e. $r_1 \geq r_j$, $j = 2,\ldots,M$. Because the number of points scored by the player at the end of the tournament, by accident, then the ideal or correct results can be expected with some probability $P_I(M,N)$ and $P_p(M,N)$, where N — the total number of games held in the tournament. Further we will concentrate on determining the probability of correctness of the tournament.

We will assume that the system of lottery A is better than the system B, if $P_p^A(M,N) > P_p^B(M,N)$.

3 The density of distribution of the number of points scored at the end of the tournament and round tournament

The j player's state is a random variable, its mathematical expectation $E_{\xi_j} = s_j$, and the variance $D_{\xi_j} = s_j(1-s_j)$. The mathematical expectation of the number of points scored by a j player in the game with a k player: $E_{r_{jk}} = 2P_{jk}(2)+1P_{jk}(1)+0P_{jk}(0)$, where $P_{jk}(2)$ - the probability of winning the j-th player, $P_{jk}(0)$ - the probability of hitting j-th player, $P_{jk}(1)$-the probability of a draw. The number of points of the k-th player $r_{jk}(1)$ in one meeting of the j-th and k-th players takes values 0,1,2 with probabilities

$$P_{jk}^1(0) = P_{kj}^1(2) = s_k(1-s_j)$$
$$P_{jk}^1(1) = P_{kj}^1(1) = 1-(s_j+s_k-2s_js_k) = 1-P_{jk}^1(0)-P_{jk}(2) \qquad (1)$$
$$P_{jk}^1(2) = P_{kj}^1(0) = s_j(1-s_k).$$

So the average number of points scored by the j-th player in the meeting with k-th player, and the variance r_{jk} are equal: $E_{r_{jk}} = 1+s_j-s_k$, $D_{r_{jk}} = s_j(1-s_j)+s_k(1-s_k)$. For the number of points $r_j < 0$ and $r_j > 2$ $P^1(r_j) = 0$. The sum of the number of points scored by both players in each game is 2. When calculating the number of points scored in the tournament the j-th player, you need to consider that he does not meet with himself, and therefore does not gain a single point. It is easiest to assume that $P_{jj}^1(0) = 1$, $P_{jj}^1(1) = P_{jj}^1(2) = 0$. In the future we will assume that the results of games do not depend on each other. Then the number of points in two meetings is equal $\nu = r_{jk}(2)$, takes the value from 0 to 4, the distribution density of this quantity is equal to the convolution $P_{jk}^2(\nu) = \sum_{\mu=0}^{2} P_{jk}^1(\mu)P_{jk}^1(\nu-\mu)$, $\nu = 0,1,2,3,4$. So, for $\nu = 0$: $P_{jk}^2(0) = (P_{jk}^1(0))^2$, for $\nu = 1$: $P_{jk}^2(1) = P_{jk}^1(0)P_{jk}^1(1) + P_{jk}^1(1)P_{jk}^1(0) = 2P_{jk}^1(0)P_{jk}^1(1)$, etc.

Taking into account (6), the following recurrence relation holds for n meetings:

$$P_{jk}^n(\nu) = \sum_{\mu=0}^{2} P_{jk}^{n-1}(\nu-\mu)P_{jk}^1(\mu), \quad \nu = 0,...,2n. \quad (2)$$

The larger n, the closer the distribution of the total number of points to the normal discrete distribution law, defined on the set of natural numbers.

Round tournament. Let M be the number of players. Total number of games $N = 0{,}5M(M-1)$, and the total number of points scored by all players $R = M(M-1)$. The number of games played by each player $n = M-1$. Since the order of the games does not affect the number of points scored, we will assume that each j-th player meets consistently with the first, second, and so on up to M. Denote the number of points received by the j-th player in all $(M-1)$ tournament games, as r_j. The number of points in each meeting is a discrete random variable having a distribution density (1). The density of the distribution of the number of points after m meetings is related to the density of distribution of the number of points after ($m-1$)-th game by a relation similar to (2):

$$P_j^m(r_j) = \sum_{\mu=0}^{2} P_j^{m-1}(r_j-\mu)P_{jm}^1(\mu) = P_j^{m-1} * P_{jm}^1, \quad r_j = 0,...,2m, \quad (3)$$

where $*$ is the sign of the convolution operation. Here it is taken into account that in the last game the j -th player meets the m -th. In accordance with (1):

$$P_{j1}^1(0) = s_1(1-s_j); \qquad P_{j1}^1(1) = 1-(s_j+s_1-2s_js_1); \qquad P_{j1}^1(2) = s_j(1-s_1). \quad (4)$$

A random variable r_j takes integer values in the range from zero to $2(M-1)$ and its distribution density is found recurrently with the initial conditions (4) by the formula (3):

$$P_j(r_j) = \sum_{\mu=r_j-2}^{r_j} P_j^{M-2}(\mu)P_{jM}^1(r_j-\mu) = \sum_{\mu=0}^{2} P_j^{M-2}(r_j-\mu)P_{jM}^1(\mu).$$

The limits of the summation are defined by the fact that the argument in the function P_{jM}^1 takes values from zero to two. The average number of points $\overline{r_j}$ scored in a one-round tournament by a j -th player is equal to the sum over $k(k \neq j)\overline{r_{jk}}$, and the variance is the same amount $D_{r_{jk}}$. So that

$$\overline{r_j} = (M-1)(1+s_j) - \sum_{k=1,k\neq j}^{M} s_k, \quad D_{r_jk} = (M-1)s_j(1-s_j) + \sum_{k=1,k\neq j}^{M} s_k(1-s_k).$$

Calculation of the probability of correctness of the tournament. Knowing $P_i(r_i)$ for all players, you need to find the probability that one player (for certainty the first) scored not less points than the j -th. The domain of probability density distribution satisfies the conditions:

$$0 \le r_i \le 2(M-1), i = 1,...,M, \ \sum_{i=1}^{M} r_i = R = M(M-1).$$

Let the first player score the maximum possible number of points *2(M-1),* then with probability one he scored points more than any j -th. The tournament is known to be correct, if at the end of the tournament $r_1 \ge 2(M-1)-1 = 2M-3$. Otherwise, when calculating the probability that the first player will score r_1 points, and at the same time j -th scored $r_j \ge r_1$, you need to take into account separately the results of a personal meeting, since these events are not independent. Denote by $\breve{P}_{1j-}(r)$ the probability distribution density of the fact that in all meetings of the 1-st player, in addition to his meeting with the j -th player, the total number of points scored is r . This probability density for $0 \le r \le 2(M-2)$ is found as follows: $\breve{P}_{1j-}(r) = P_{12}(r) * ... * P_{1(j-1)}(r) * P_{1(j+1)}(r) * ... * P_{1M}(r),$

where $P_{ik}(r)$ is the probability distribution density of the i-th player receiving r points in the game with the k-th player. The symbol * is used to denote the convolution operation:

$$p_1(r) * p_2(r) = \sum_{r_0=0}^{2} p_1(r_0)p_2(r-r_0).$$

Similarly founded the distribution density of points $\breve{P}_{j1-}(r)$ scored in all games by the j -th player, except for the player's game with the first: $\breve{P}_{j1-}(r) = P_{j3}(r) * P_{j4}(r) * P_{j5}(r) * ... * P_{jM}(r).$

Consider three possible outcomes of a meeting between the 1st and the j -th players.

A) Player 1 has lost to the j -th player. In this case, the probability $P_1(r)$ is equal to $P_1(r) = \breve{P}_{1j-}(r)$. B) Player 1 played a draw with the player *j*: $P_1(r) = \breve{P}_{1j-}(r-1)$. C) Player 1 has won from the player *j*: $P_1(r) = \breve{P}_{1j-}(r-2)$.

Let's find $P(r_1 > r_j)$ the probability that the number of points r_1 scored by the 1-st player will not be less than the number of points r_j of the j -th player. Let's write out the expressions for this probability, depending on the results of the personal meeting of the 1-st and the j -th players:

A) $P_a(r_1 \geq r_j) = \sum_{r_1=2}^{2(M-2)} \left(\breve{P}_{1j-}(r_1) \sum_{r_j=2}^{r_1} \breve{P}_{j1-}(r_j - 2) \right)$ B) $P_b(r_1 \geq r_j) = \sum_{r_1=1}^{2(M-2)+1} \left(\breve{P}_{1j-}(r_1 - 1) \sum_{r_j=1}^{r_1} \breve{P}_{j1-}(r_j - 1) \right)$

C) $P_c(r_1 \geq r_j) = \sum_{r_1=0}^{2(M-2)+2} \left(\breve{P}_{1j-}(r_1 - 2) \sum_{r_j=0}^{r_1} \breve{P}_{j1-}(r_j) \right)$

The probability $P(r_1 \geq r_j)$ that the first player in the final table will be not lower than the j -th is found as a weighted average of the recorded values, taking into account the probabilities of the results of the personal meeting as follows: $P(r_1 \geq r_j) = P_{1j}(0)P_a(r_1 \geq r_j) + P_{1j}(1)P_b(r_1 \geq r_j) + P_{1j}(2)P_c(r_1 \geq r_j)$, where $P_{1j}(r)$ is the probability distribution density of the fact that the 1-st player will score r points in the game with the j -th player. However, to calculate the probability of correctness of the tournament, you cannot use the product of the probabilities $P(r_1 \geq r_j)$ that the first player will score more points than the j -th, for $j = 2,3,...,M$, since each of these probabilities depends on whether the first player scored more points than the k -th. The statement is true: *The tournament is correct, if for any possible number of points* r_1 *scored by the first player, none of the other players scored more points*. The minimum number of points that the first player must score in order to divide the first place with a probability greater than zero is $M-1$. In this case, all games will end in a draw and he will share the first place with the other participants. If, as mentioned above, he collects a $2M-3$ point, he will certainly be the first or will share the first place with one of the other participants of the tournament. So the probability of a correct tournament in this case is one.

It should be noted that the total number of points scored by all participants at the end of the tournament is fixed and equal $M(M-1)$. Hence, if the first player scored r_1 points, then the remaining players in the total will gain $M(M-1)-r_1$ points. The probability that all players in the sum will gain a certain number of points r_σ, will be denoted by P_σ. It is equal to the convolution of the densities of the distribution of the number of points scored according to the results of the tournament by the second, third, etc., M -th players: $P_\sigma(r_\sigma) = P_2(r_2) * P_3(r_3) * ... * P_M(r_M)$.

The probability of correctness of the tournament in the general case can be expressed by the formula: $P_p = \sum_{r_1=M-1}^{2(M-1)} P_1(r_1) \sum_{r_j \in V, j=2...M} P_2(r_2)P_3(r_3)...P_M(r_M)$. (5)

In this case, the set of admissible values V of the arguments $r_j, j = 2,3,...,M$ is determined by the constraints: $\sum_{j=2}^{M} r_j = M(M-1) - r_1, \quad r_j \leq r_1 \forall j.$ Practically using the formula (5), requires a cumbersome search and when $M > 10$ it becomes too time-consuming computation. Below are the formulas that allow you to get an estimate of the probability of correctness of the tournament with much less labor. Since $r_\sigma = M(M-1) - r_1$, then the probability P_σ is determined for a fixed number of players for each value r_1. Therefore, bearing in mind such a substitution, we will write $P_\sigma(r_1)$. To assess the probability of correctness of the tournament, we obtain expression:

$$P_p(M,N) = \sum_{r_1=M-1}^{2M-3} P_1(r_1) \prod_{i=2}^{M} \left[1 - \sum_{r_i=r_1+1}^{2M-3} P_i(r_i) \right] P_\sigma(r_1). \quad (6)$$

Here random events "no player other than the first will score more points than r_1" and "all players except the first score in the sum r_σ points" are supposed to be independent. Therefore, the evaluation of the correctness of the tournament found in this way is higher than the true value found by the formula (5), but the calculation does not require busting. To estimate the accuracy of the last formula, we calculate the error of estimating probability $P_p(M,N)$ found by the formula (6), with respect to the probability P_p, obtained by the formula (5), for different a priori distribution of skills $s_1, s_2, ..., s_M$, using the following expression: $\Delta\% = \frac{P_p(M,N) - P_p}{P_p} 100\%$

These calculations showed that the estimate (6) has an error of the order of 4%, which allows it to be used to compare the different schemes of the tournament.

4 Conclusions

A criterion is proposed for evaluating the way a tournament is organized as the maximum probability of winning an objectively strongest player. Calculation formulas for the probability distribution density of the number of points scored in the tournament by one or the other player are obtained. An analytical estimation of the probability of the tournament results is proposed, which in many cases allows avoiding the laborious procedure of enumeration of the admissible variants. All the results obtained refer to the task of determining the best element of the sample by means of an examination in which the expert chooses the best by means of a series of paired comparisons [4].

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